Journal homepage www.jzs.univsul.edu.iq
Journal of Zankoy Sulaimani
Part-A- (Pure and Applied Sciences)

Using Optimal Geometric Average Technique to Solve Extreme Point Multi- Objective Quadratic Programming Problems

Nejmaddin A. Sulaiman[1], Ronak M. Abdullah[2] and Snur O. Abdull[2]

1 Mathematical Department, College of Education/ Salahaddin University -Erbil.
2 Mathematics Department, School of Science /University of Sulaimani.

Article info

Original:
02 October 2015
Revised:
25January 2016
Accepted:
28 January 2016
Published online:
20 September 2016

Key Words: Optimal geometric average
EPMOQPP
Cutting plane technique

Abstract

In this paper, we suggested a new technique by using optimal geometric average for the value of functions, to solve extreme point multi- objective quadratic programming problem (EPMOQPP), via transforming it to extreme point single-objective quadratic programming problem(EPSOQPP), then solve the problem by Wolfe's method [1] , and an algorithm is given for its solution, also using cutting plane technique when the optimal value of the objective function is not an extreme point of constraints, the computer application for this algorithm was tested on a number of numerical examples, which are also solved by several techniques (Chandra Sen.'s, mean, median, and optimal arithmetic). After comparing, the results indicate that the new technique is best than others as shown in table (3).

1. Introduction

In this paper, we used optimal geometric average technique for solving extreme point multi-objective quadratic programming problems. Following indicate some researchers work in this field.

In (1983) Sen., Ch., studiedA new approach for multi-objective rural development planning, The India Economic Journal [3], (1988) Al-Barzinji, S. H. studied extreme point optimization technique in mathematical programming [6], (1989) Sulaiman, N. A. discussed extreme point of the quadratic programming problem technique [13], (2006) Sulaiman, N. A. and Sadiq, G. W. studied solving the multi-objective linear programming problem using mean and median value [5], (2011) Abdul Rahim, B. K. studied solving quadratic programming with extreme point [9], (2011) Sulaiman, N., A., and Abdul-Rahim, B., K., studied optimal cutting plane procedure for MOQPP, Jornal of koya University [4], (2013) Sulaiman, N.A. and Abulrahim, B. K. studied arithmetic average transformation technique to solve multi-objective quadratic programming [2], (2014) Sulaiman, N., A., and Abdul-Rahim, B., K., studied New arithmetic average technique to solve Multi-objective linear fractional programming problem and its comparison with other techniques [7], (2015) Sulaiman, N., A., and Nawkhass, M. A. using short-hierarchical method to solve MOLFPP [8].

2. Some basic definitions:

This section consists of some definitions which are needed in this paper.

2.1 Quadratic programming (QP):

A quadratic programming (QP) is a special type of mathematical optimization problem. It is the problem of optimizing (minimizing or maximizing) a quadratic function of several variables subject to linear constraints on these variables.

The quadratic programming problem can be formulated as follows. [10]

Suppose n is a positive integer representing the number of variables and m is a positive integer representing the number of constraints. Suppose c is an n-dimensional real vector, Q is an $n \times n$ real symmetric matrix, A is an $m \times n$ real matrix, , x is the n-dimensional column vector decision variables and b is an m-dimensional real vector.

The quadratic programming problem is:

$$\text{Optimize } f(x) = \frac{1}{2}x^T Q x + c^T x$$

Subject to the linear constraints:

$$Ax \begin{bmatrix} \leq \\ = \\ \geq \end{bmatrix} b \qquad \dots (1)$$

$$x \geq 0.$$

2.2 Geometric average:

In mathematics, the geometric mean is a type of mean or average, which indicates the central tendency or typical value of a set of numbers by using the product of their values (as opposed to the arithmetic mean which uses their sum). The geometric mean is defined as the n^{th} root of the product of n numbers. [11]

The mathematical form of Geometric Average is:

Geometric Average = $\sqrt[n]{\prod_{i=1}^{n} x_i}$

Where x = Individual score and n = Sample size (Number of scores)

2.3 Feasible solution:

A feasible solution to a linear program problem is a solution that satisfies all constraints. [12]

2.4 Extreme point of the QPP:

The extreme point of the quadratic programming problem can be defined as follows:

$$Max. f = c^T x + \frac{1}{2}x^T Q x$$

Subject to:-

$$Ax = b \dots (2)$$

Such that x is an extreme point of other constraint set

$$Hx = h$$

$$x \geq 0$$

Where A is$(m \times n)$ matrix of coefficients, and x, c are $(n \times 1)$vector of constant, b is $(m \times 1)$ vector, Q is a symmetric square matrix, H is$(p \times n)$ matrix of constant, h is $(p \times 1)$ vector, the transpose of the vector denoted by (T).[14]

3. Multi-objective quadratic programming problem (MOQPP):

The mathematical form of multi-objective quadratic programming problems is:

$$\left.\begin{aligned} \text{Max. } f_1 &= \tfrac{1}{2}x^T Q_1 x + C_1^T x \\ \text{Max. } f_2 &= \tfrac{1}{2}x^T Q_2 x + C_2^T x \\ &\vdots \\ \text{Max. } f_r &= \tfrac{1}{2}x^T Q_r x + C_r^T x \\ Min.\, f_{r+1} &= \tfrac{1}{2}x^T Q_{r+1} x + C_{r+1}^T x \\ Min.\, f_{r+2} &= \tfrac{1}{2}x^T Q_{r+2} x + C_{r+2}^T x \\ &\vdots \\ Min.\, f_s &= \tfrac{1}{2}x^T Q_s x + C_s^T x \end{aligned}\right\} \dots (3)$$

Subject to:

$$Ax \begin{bmatrix} \le \\ = \\ \ge \end{bmatrix} b \qquad \dots (4)$$

$x \geq 0$… (5)

Where Q is an $(n \times n)$ symmetric matrix of coefficients, x is an n–dimensional vector of decision variables, C is the n–dimensional vector of constants, b is m-dimensional vector of constants,A is $(m \times n)$ matrix of coefficients, r is number of objective functions to be maximized, s is the number of objective functions to be maximized and minimized and $(s - r)$ is the number of objective functions that is minimized,all vectors are assumed to be column vectors unless transposed(T), $i = 1,2, \dots s$.

4. Extreme Point Multi-Objective Quadratic Programming Problem (EPMOQPP)

The extreme point of multi-objective quadratic programming problem can be defined as follows:

$$\left.\begin{aligned} Max.\, f_1 &= C_1^T x + \tfrac{1}{2}x^T Q_1 x \\ Max.\, f_2 &= C_2^T x + \tfrac{1}{2}x^T Q_2 x \\ &\vdots \\ Max.\, f_r &= C_r^T x + \tfrac{1}{2}x^T Q_r x \\ Min.\, f_{r+1} &= C_{r+1}^T x + \tfrac{1}{2}x^T Q_{r+1} x \\ &\vdots \\ Min.\, f_{s=} & C_s^T x + \tfrac{1}{2}x^T Q_s x \end{aligned}\right] \dots (6)$$

Subject to:-

$Ax = b$… (7)

Such that x is an extreme point of other constraint set

$Hx = h$ … (8)

$x \geq 0 \quad \dots (9)$

For solving the problem of equation (6) the problem become as follows:

$$\left.\begin{array}{c} Max. f_1 = C_1^T x + \frac{1}{2} x^T Q_1 x \\ Max. f_2 = C_2^T x + \frac{1}{2} x^T Q_2 x \\ . \\ . \\ . \\ Max. f_r = C_r^T x + \frac{1}{2} x^T Q_r x \\ Min. f_{r+1} = C_{r+1}^T x + \frac{1}{2} x^T Q_{r+1} x \\ . \\ . \\ . \\ Min. f_{s=} C_s^T x + \frac{1}{2} x^T Q_s x \end{array}\right] \dots (10)$$

Subject to:-

$Lx = l \dots (11)$

And $\quad x \geq 0 \qquad \dots (12)$

Where $L = \binom{A}{H}$ is of order$(p \times m) \times n$, $l = \binom{a}{h}$is of order$(p \times m) \times 1$.

Notation used, Let $E_1 = \{x: Ax = b$ and x is an extreme pint of$Hx = h$, $x \geq 0\}$

And$E_2 = \{x: x$ is an extreme pint of $Lx = l, x \geq 0\}$.

5. Formulation of (EPMOQPP):

Suppose we obtain a single value corresponding to each of the objective functions of the MOQPP of the equation (3).They are being optimized individually subject to the constraints (4) and (5) as follows:

$$\begin{array}{l} Max. f_1 = \theta_1 \\ Max. f_2 = \theta_2 \\ . \\ . \\ Min. f_r = \theta_r \\ Min. f_{r+1} = \theta_{r+1} \\ . \\ . \\ Min. f_s = \theta_s \end{array}$$

Where $\theta_i \neq 0$, $i = 1,2, \dots s$ are values of the objective functions, the decision variable may not necessarily be common to all optimal solutions in the presence of conflicts among objectives. But the common sets of decision variable between objective functions are necessary in order to select the best compromise solution. Applied Chandra Sen.'s technique to solve EPMOQPP, which is of the form:

$Max. f = \sum_{i=1}^{r} \frac{f_i}{|\theta_i|} - \sum_{i=r+1}^{s} \frac{f_i}{|\theta_i|} \dots (13)$

And solve it by Wolfe's method with the same constraints (11) & (12)

6. Solving EPMOQPP by using Optimal Geometric Average technique:

We can formulate the combined objective function (10) as follows to determine the common set of decision variables. Here using Optimal Geometric Average technique for solving EPMOQPP as follows:

$Max. f = \frac{\sum_{i=1}^{r} f_i - \sum_{r+1}^{s} f_i}{OG_v}$… (14)

Subject to the same constraints (11), (12)

Where OG_v denotes the optimal geometric average, and calculated as bellow:

$m_1 = min(OM_i)$, where$(OM_i) = |\theta M_i|$, and θM_i is the maximum value off_i, $\forall\ i = 1,2, \dots, r$

$m_2 = min(ON_i)$, where$(ON_i) = |\theta N_i|$, and θN_i is the minimum value off_i, $\forall\ i = r+1, r+2, \dots, s$

$$OG_v = \begin{Bmatrix} \frac{\sqrt[r]{n_1} + \sqrt[s-r]{n_2}}{2}, & where\ s - r \geq 2 \\ \frac{\sqrt[r]{n_1}}{2}, & where\ s = r \\ \frac{\sqrt[s-r]{n_2}}{2}, & where\ s = s - r \end{Bmatrix}$$

Special case: A special case will appear when one of the optimal value is equal to zero. In this case the algorithm does not work.

6.1 Algorithm for solving EPMOQPP by OptimalGeometric Average technique

We can summarized the following algorithm to get the optimal solution for the optimal CPMOQPP as follows:

Step 1: Assign arbitrary values to each of the individual objective functions which are to be maximized and minimized for problem (10).

Step 2: Solve each of the objective functions (10) by the Wolfe's method [1], for quadratic programming problems subject to the constraints (11), (12).

Step 3: Check the solutions for feasibility in step 2, if the solutionsare feasible go to step 4, otherwise, use dual wolf's method [1], to remove infeasibility.

Step 4: Set a name to the solution values of each objective functions $f_1, f_2, \dots, f_s$, namely $\theta_1, \theta_2, \dots, \theta_s$, respectively**.**

Step 5: Select $n_1 = \min(OM_i), i = 1,2, \dots, r$

$n_2 = \min(ON_i), i = r+1, r+2, \dots, s$

And calculate $OG_v = \begin{Bmatrix} \frac{\sqrt[r]{n_1} + \sqrt[s-r]{n_2}}{2}, & where\ s - r \geq 2 \\ \frac{\sqrt[r]{n_1}}{2}, & where\ s = r \\ \frac{\sqrt[s-r]{n_2}}{2}, & where\ s = s - r \end{Bmatrix}$, $S_1 = \sum_{i=1}^{r} f_i$and$S_2 = \sum_{i=r+1}^{s} f_i$.

Step 6: Construct the combined objective function which has the formula (14).

Step 7: Optimize objective function that became in step 6 byWolfe's method [1] under the same constraints (11) and (12).

Step 8: If the solution that became in step 7 intersecting by E_1 is not empty thus; terminate, otherwise go to step 9.

Step 9: Using cutting plane technique by adding a new constraint which is constructed by this formula

$\sum_{i=1}^{r} C_i^T x - \sum_{i=r+1}^{s} C_i^T x \leq \frac{\sum_{i=1}^{s} |f_i|}{s}$, where, $C_i^T x$ is a linear part of the maximum and minimum objective functions.

Step 10: Again optimize objective function that became in step 6 by Wolfe's method [1] under the new constraint that we get in step 9 and the constraints (11) and (12).

6.2 Numerical results and Examples:

In this section we constructed some numerical Examples (EPMOQPP) and solved them. But only two of them presented here.

Example 6.2.1: Solve the following EPMOQPP

$$Max. f_1 = -3x_1^2 - 4x_1x_2 - 3x_2^2 + 5x_1 + 3x_2 + 50$$
$$Max. f_2 = -3x_1^2 - 6x_1x_2 - 3x_2^2 + 4x_1 + 2x_2 + 45$$
$$Max. f_3 = -5x_1^2 - 10x_1x_2 - 4x_2^2 + 10x_1 + 10x_2 + 60$$
$$Min. f_4 = x_1^2 + 2x_1x_2 + x_2^2 - 2x_1 - 2x_2 - 40$$
$$Min. f_5 = 7x_1^2 + 14x_1x_2 + 7x_2^2 - 14x_1 - 10x_2 - 44$$
$$Min. f_6 = 8x_1^2 + 16x_1x_2 + 8x_2^2 - 16x_1 - 12x_2 - 43$$

Subject to:-

$$2x_1 + x_2 \geq 2$$

And (x_1, x_2) is an Extreme point of:

$$3x_1 + 4x_2 \leq 12$$
$$2x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution:

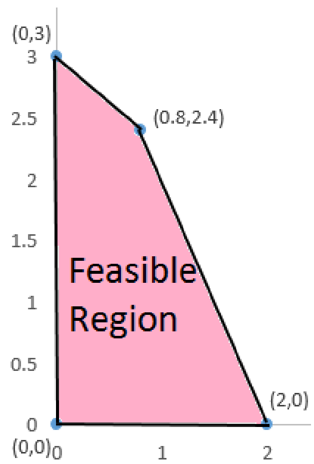

Figure (6.2.1.1): Feasible Region of Example (6.2.1) before cuttingPlane

Solve each of objective function with the same constraint by Wolfe's method [1], we get

Table 1: Solution of each $f_i$of Example 6.2.1

i	f_i	x_i	θM_i	OM_i=$\lvert\theta M_i\rvert$	θN_i	OM_i=$\lvert\theta M_i\rvert$
1	52	(1,0)	52	52		
2	46		46	46		
3	65		65	65		
4	-41				-41	41
5	-51				-51	51
6	-51				-51	51

First using **the Optimal Geometric Average (OG_v) Technique** to solve Example 6.2.1 by using formula (14)

Find $n_1 = \min(OM_i) = 46 \rightarrow \sqrt[3]{n_1} = \sqrt[3]{46} = 3.583048.$

And $n_2 = \min(ON_i) = 41 \rightarrow \sqrt[3]{41} = 3.448217.$

Calculate$OG_v = \frac{\sqrt[r]{n_1} + \sqrt[s-r]{n_2}}{2} = 3.515632$

$$S_1 = \sum_{i=1}^{3} f_i = -11x_1^2 - 20x_1x_2 - 10x_2^2 + 19x_1 + 15x_2 - 155$$

$$S_2 = \sum_{i=1}^{3} f_i = 16x_1^2 + 32x_1x_2 + 16x_2^2 - 32x_1 - 24x_2 - 127$$

$$Max.f = \frac{S_1 - S_2}{OG_v}$$

$$= \frac{-27x_1^2 - 52x_1x_2 - 26x_2^2 + 51x_1 + 39x_2 - 282}{3.515632}$$

$$= -7.679985x_1^2 - 14.791082x_1x_2 - 7.395541x_2^2 + 14.506638x_1 + 11.093311x_2 + 80.213174$$

So,

$Max.f = -7.679985x_1^2 - 14.791082x_1x_2 - 7.395541x_2^2 + 14.506638x_1 + 11.093311x_2 + 80.213174$

Subject to:

$$2x_1 + x_2 \geq 2$$
$$3x_1 + 4x_2 \leq 12$$
$$2x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solve it by Wolfe's method [1] we get

$Max.f = 87.039827$at (1,0), but (1,0)is not an extreme point of:

$$3x_1 + 4x_2 \leq 12$$
$$2x_1 + x_2 \leq 4$$

So, using cutting plane technique by adding the new constraint by this formula

$\left(\sum_{i=1}^{r} C_i^T x - \sum_{i=r+1}^{s} C_i^T x \leq \frac{\sum_{i=1}^{s} f_i}{s}\right)$, we get

$$51x_1 + 39x_2 \leq \frac{306}{6}$$
$$51x_1 + 39x_2 \leq 51$$

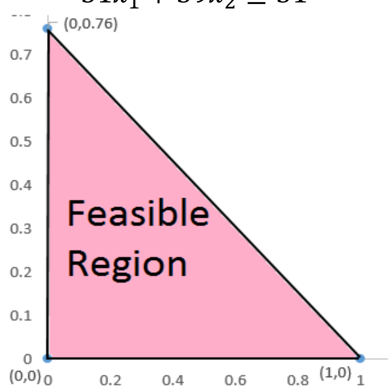

Figure 6.2.1.2: Feasible Region while using cutting plane Example 6.2.1

Thus we get

$$Max.f = -7.679985x_1^2 - 14.791082x_1x_2 - 7.395541x_2^2 + 14.506638x_1 + 11.093311x_2 + 80.213174$$

Subject to:

$$2x_1 + x_2 \geq 2$$
$$3x_1 + 4x_2 \leq 12$$
$$2x_1 + x_2 \leq 4$$
$$51x_1 + 39x_2 \leq 51$$
$$x_1, x_2 \geq 0$$

Solve it again by Wolfe's method [1], become
$Max.f = 87.039827$at (1,0), and (1,0)is an extreme point of:

$$3x_1 + 4x_2 \leq 12$$
$$2x_1 + x_2 \leq 4$$
$$51x_1 + 39x_2 \leq 51$$

And feasibility of:-

$$2x_1 + x_2 \geq 2$$

Using **Chandra Sen.'s (CA)Technique[3]**to solve Example 6.2.1 the result is $Max.f = 5.862744$ at (1,0)
Using **modified (mean and median) (MA)Technique [5]**to solve Example 6.2.1 we get, $Max.f = 6.000001$ at (1,0) by Mean technique and by Median technique the result is $Max.f = 6.243311$ at(1,0),
Using **optimal arithmetic average (OA_v)technique [2]**to solve Example 6.2.1, the result is $Max.f = 7.034483$ at(1,0).

Example 6.2.2: Solve the following EPMOQPP

$$Max.f_1 = -x_1^2 - 2x_1x_2 - 2x_2^2 + 6x_1 + 10x_2 + 134$$
$$Max.f_2 = -2x_1^2 - 2x_1x_2 - 3x_2^2 + 8x_1 + 14x_2 + 126$$
$$Min.f_3 = 2x_1^2 + 4x_1x_2 + 2x_2^2 - 12x_1 - 12x_2 - 100$$
$$Min.f_4 = x_1^2 + 4x_1x_2 + 2x_2^2 - 12x_1 - 14x_2 - 120$$

Subject to:-

$$x_1 + x_2 \leq 3$$

And (x_1, x_2) is an extreme point of

$$2x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution:-

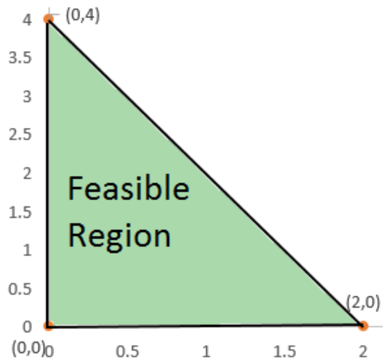

Figure (6.2.2.1): Feasible region of Example (6.2.2) before cutting

Solve each of objective function with the same constraint by Wolfe's method [1], we get

Table 2: Solution of each $f_i$of Example 6.2.2

I	f_i	x_i	θM_i	OM_i=$\lvert\theta M_i\rvert$	θN_i	OM_i=$\lvert\theta M_i\rvert$
1	147	(1,2)	147	147		
2	144		144	144		
3	-118				-118	118
4	-143				-143	143

First using the **Optimal Geometric Average(OG_v)Technique**to solve Example 6.2.2 by using formula (14) the result is $Max.f = 48.28809$ at (1,2), but (1,2)is not an extreme point of:

$$2x_1 + x_2 \leq 4$$

So, using cutting plane technique by adding new constraint

$$38x_1 + 50x_2 \leq \frac{552}{4} = 138$$

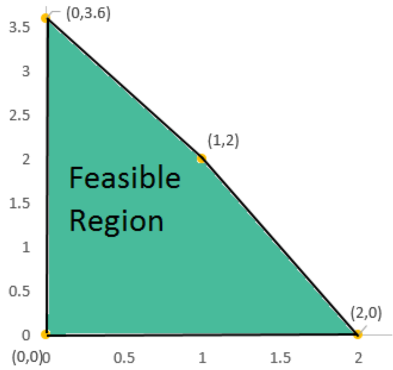

Figure (6.2.2.2): Feasible region of Example (6.2.2) using cuttingplane

Thus we get $Max.f = 48.28809$at (1,2), and (1,2)is an extreme point of:

$$2x_1 + x_2 \leq 4$$
$$38x_1 + 50x_2 \leq 138$$

And feasibility of:

$$x_1 + x_2 \leq 3$$

Using **Chandra Sen.'s (CA) Technique[3]** to solve Example 6.2.2 the result is $Max.f = 4.000004$ at(1,2), Using **modified (mean and median) (MA)Technique [5]**to solve Example 6.2.2 we get $Max.f = 4.000003$ at(1,2),

Using **optimal arithmetic average (OA_v) technique [2]** to solve Example 6.2.2 the result is $Max.f = 4.213731$ at(1,2).

Table 3: Compare between results obtained by (OG_v), (CA), (MA)& (OA_v) Techniques

Examples	Chandra Sen.'s Technique	Modified Technique		Optimal Arithmetic Average Technique	Optimal Geometric Average Technique
		Using Mean	Using Median		
Ex. 6.2.1	$Max.f = 5.862744$ $x_1 = 1, x_2 = 0$	$Max.f = 6.000001$ $x_1 = 1, x_2 = 0$	$Max.f = 6.243311$ $x_1 = 1, x_2 = 0$	$Max.f = 7.034483$ $x_1 = 1, x_2 = 0$	$Max.f = 87.039827$ $x_1 = 1, x_2 = 0$
Ex. 6.2.2	$Max.f$ =4.000004 $x_1 = 1, x_2 = 2$	$Max.f = 4.000003$ $x_1 = 1, x_2 = 2$	$Max.f = 4.000003$ $x_1 = 1, x_2 = 2$	$Max.f = 4.213731$ $x_1 = 1, x_2 = 2$	$Max.f = 48.28809$ $x_1 = 1, x_2 = 2$

7. Conclusion:

The new technique used (optimal geometric average) to solve EPMOQPP. In the proposed approach, transform extreme point multi-objective quadratic programming problem into extreme point single-objective quadratic programming problem by optimal geometric average technique; then Wolfe's method [1] used to solve the problem; and for getting the optimal value in one of the extreme point for constraints used cutting plane technique. The results obtained by this method have a good agreement with the results that obtained by other methods. The main advantage of the proposed technique is better than other techniques as shown in table (3).

References:

[1] Wolfe, P.,"*The simplex method for quadratic programming*", Econometrical, Vol.27, pp. 382-389, (1959).

[2] Sulaiman, N.A. and Abulrahim, B. K., "*Arithmetic average transformation technique to solve multi-objective quadratic programming",* Vol.15, No.1, (2013).

[3] Sen., Ch.,"*A new approach for multi-objective rural development planning*", The India Economic Journal, Vol. 30, No. 4, pp.91-96, (1983).

[4] Sulaiman, N., A., and Abdul-Rahim, B., K., "*Optimal cutting plane procedure for MOQPP",* Jornal of koya University, ISSN.2073-20713, No-20, pp.119-130, (2011).

[5] Sulaiman, N. A. and Sadiq, G. W., "*studied solving the multi-objective linear programming problem using mean and median value*", Al-Rafiden Journal of computer sciences and mathematics, mosul university, Vol.3, No.1, pp. 69-83(2006).

[6] Al-Barzinji, S. H. A., "*Extreme Point Optimization Technique in Mathematical Programming*", M.Sc. Thesis, University of Salahaddin, Erbil/ Iraq(1988).

[7] Sulaiman, N., A., and Abdul-Rahim, B., K., "*studied New arithmetic average technique to solve Multi-objective linear fractional programming problem and its comparison with other techniques*",IJRRAS, Vol.18, No.2, (2014).

[8] Sulaiman, N., A., and Nawkhass, M. A., "*using short-hierarchical method to solve MOLFPP*", 2nd scientific conference of Garmian University, may/6,7(2015).

[9] Abdul Rahim, B. K., "*Solving quadratic programming with extreme point*", MSc. Thesis, Sulaimani University, Sulaimani-Iraq (2011).

[10] http://en.wikipedia.org/wiki/Quadratic_programming#cite_note-1

[11]http://en.wikipedia.org/wiki/Geometric_mean

[12] http://www.columbia.edu/~cs2035/courses/ieor4600.S07/lec1.pdf

[13] Sulaiman, N. A., "*Extreme Point Quadratic Programming Problem Techniques*", M.Sc.,Thesis, University of Salahaddin, Hawler\Iraq (1989).

[14] Kirby M. J., love H. R. and Swarup K., "*Extreme point programming with non-linear constraints*", discrete mathematics, Vol. 15, pp. 345-365 (1973).